SOLUTION OF THE MONO-ENERGETIC NEUTRON-TRANSPORT EQUATION BY RATIONAL FUNCTION APPROXIMATION AND ITS APPLICATIONS

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of

MASTER OF TECHNOLOGY

by

C. K. VENKATESAN

to the

DEPARTMENT OF NUCLEAR ENGINEERING AND TECHNOLOGY PROGRAMME
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

MAY, 1985

CERTIFICATE

This is to certify that this work on ''SOLUTION OF MONO-ENERGETIC NEUTRON TRANSPORT EQUATION BY A RATIONAL FUNCTION APPROXIMATION AND ITS APPLICATIONS'' by Mr. C.K. VENKATESAN has been carried out under my supervision and has not been submitted elsewhere for the award of a degree.

Dr. A. SENGUPTA
ASSISTANT PROFESSOR
NUCLEAR ENGINEERING AND
TECHNOLOGY PROGRAMME,
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

ACKNOWLEDGEMENTS

I express my deep sense of gratitude to Dr. A. Sengupta for his valuable guidance and critical appraisal throughout the work. My sincere thanks are also due to the faculty and staff of Nuclear Engineering Dept. for their timely help.

I also express my sincere thanks to Mr. Sundrarajan, Mr. Shivakumar, Mr. Veluswamy, Mr. Sivasubramaniam, Mr. Sekar and Mr. Pandimani for their invaluable advice and timely help. A big thanks to all my friends who made my stay in IIT Kanpur pleasant and memorable.

Finally thanks are for my parents, sister and muthu for their love and encouragement during my stay away from home, that kept me going.

Special thanks are due to Mr. S.K. Tewari for his good typing work.

May 1985 - C.K. VENKATESAN

ABSTRACT

A simple accurate weight-function $\bar{W}(\mu)$ has been used instead of the exact $W(\mu)$ and with this weight-function orthogonal polynomials C_i have been developed. The solution of the one-speed transport equation has been found out by taking the transient integral as a sum and using a rational function approximation $\varphi_\varepsilon(\nu,\mu)$ to the exact $\varphi(\nu,\mu)$. This method of finding out the solution of the transport equation by approximation has been applied to different problems of the transport theory and the results are found to be reliable.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

The one-speed transport equation plays an important role in the transport theory. The complete set of eigenfunctions of this equation contains singular distributions in the form of the Cauchy principal value and dirac delta ~~functionals~~ distributions besides regular functions, see Case and Zweifel [1] for details.

We are concerned with the solution of the homogeneous one-speed equation

$$\mu \frac{\partial \Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \Psi(x,\mu')d\mu' \qquad (1.1)$$

for full-and half-range boundary conditions. There are two different ways of obtaining the solution of this equation. One is the Case's method and the other is Wiener-Hopf method. The Case's method, like the Wiener-Hopf method, is ~~accurate~~ exact and gives the exact solution of Eq. (1.1). It is analogous in some respects to the method of separation of variables commonly used for the solution of partial differential equations. In both, a complete set of elementary basis solutions is sought, and then a suitable combination of solutions is found that will satisfy the boundary conditions or the conditions at the source. The major

difference is that one of the elementary solutions of the transport equation is singular, i.e., it has meaning only when it appears inside integrals.

The Case's solution in standard notation [1] is

$$\Psi(x,\mu) = a_{0+} \exp(-x/\nu_0)\, \varphi_{0+}(\mu) + a_{0-} \exp(x/\nu_0)\, \varphi_{0-}(\mu) + \int_{-1}^{1} A(\nu) \exp(-x/\nu) \varphi_\nu(\mu) d\nu \qquad (1.2a)$$

and

$$\Psi(x,\mu) = a_{0+} \exp(-x/\nu_0) \varphi_{0+}(\mu) + \int_{0}^{1} A(\nu) \exp(-x/\nu) \varphi_\nu(\mu) d\nu$$

$$\mu \geq 0, \quad x \geq 0 \qquad (1.2b)$$

for the full-and half-range problems, respectively.

The Case eigen-functions are

$$\varphi_{0\pm}(\mu) = \frac{c\nu_0}{2} \frac{1}{\nu_0 \mp \mu} \qquad (1.3)$$

$$\varphi_\nu(\mu) = \frac{c\nu}{2} P \frac{1}{\nu - \mu} + \lambda(\nu) \delta(\nu - \mu) \qquad (1.4)$$

where ν_0 satisfies

$$\frac{c\nu_0}{2} \ln \frac{\nu_0 + 1}{\nu_0 - 1} = 1 \qquad (1.5)$$

The necessary and sufficient conditions for the determination of the constants $a_{0\pm}$ and $A(\nu)$ are the orthogonality integrals [1]

$$\int_{-1}^{1} \mu\, \varphi^2_{o\pm}(\mu)d\mu = N_{o\pm} \tag{1.6a}$$

$$\int_{-1}^{1} \mu\varphi_\nu(\mu)\, \varphi_{\nu'}(\mu)d\mu = N(\nu)\delta(\nu-\nu') \quad \nu,\nu' \varepsilon (-1,1) \tag{1.6b}$$

where

$$N_{o\pm} = \pm \frac{c\nu_o^3}{2} \left(\frac{c}{\nu_o^2-1} - \frac{1}{\nu_o^2}\right) \tag{1.6c}$$

$$N(\nu) = \nu\left[\left(1- \frac{c\nu}{2} \ln \frac{1+\nu}{1-\nu}\right)^2 + \frac{c^2\pi^2\nu^2}{4} \right] \tag{1.6d}$$

for the full-range problem, and when $\nu, \nu' \; \varepsilon \; (0,1)$,

$$\int_{0}^{1} W(\mu)\, \varphi_{o+}(\mu)\, \varphi_\nu(\mu)d\mu = 0 \tag{1.7a}$$

$$\int_{0}^{1} W(\mu)\, \varphi_\nu(\mu)\, \varphi_{\nu'}(\mu)d\mu = W(\nu)(N(\nu)/\nu)\delta(\nu-\nu') \tag{1.7b}$$

$$\int_{0}^{1} W(\mu)\, \varphi_{o\pm}(\mu)\, \varphi_{o+}(\mu)d\mu = \mp \left(\tfrac{1}{2}\, c\nu_o\right)^2 X(\pm\, \nu_o) \tag{1.7c}$$

for the half-range case [1].

The set of eigen-functions $\{\varphi_{o\pm}(\mu),\ \varphi_\nu(\mu),\ -1 \leq \nu \leq 1\}$ of Eq. (1.1) are complete in $L_2[0:1]$ with respect to the weight function $W(\mu)$ given by

$$W(\mu) = \frac{c\mu}{2(1-c)}\ \frac{1}{(\nu_o+\mu)X(-\mu)} \tag{1.8}$$

where

$$X(-\mu) = \frac{1}{\alpha+\mu}\ \Omega(-\mu) \tag{1.9}$$

$$\Omega(-\mu) = 1 - \frac{c\nu_o^2}{2}\mu \int_o^1 \frac{1-t^2X^2(0)}{(\nu_o^2-t^2)(\mu+t)\Omega(-t)}\,dt \qquad (1.10)$$

with

$$X(0) = \frac{1}{\nu_o\sqrt{1-c}} \qquad (1.11)$$

Also

$$X(\nu_o) = -\left(\frac{\nu_o^2(1-c)-1}{2a_{oM}(1-c)\ \nu_o^2(\nu_o^2-1)}\right)^{1/2} \qquad (1.12)$$

$$X(-\nu_o) = a_{oM}\ X(\nu_o) \qquad (1.13)$$

where

$$a_{oM} = -\exp(-2z_o/\nu_o)\ , \qquad (1.14)$$

z_o being the extrapolated and point. Once the coefficients are determined, then the flux and leakage can be calculated according to the problem considered. The weight function $W(\mu)$ given by Eq. (1.8) is a non-analytical function. Also the weight function is expressed in terms of $\Omega(-\mu)$ which satisfies the non-linear integral equation (1.10). Though the Case's method solution is exact, it is quite complicated to evaluate the integral in the solution of the transport equation because of its singular nature and the nature of $A(\nu)$. For example, consider the Milne problem [1] in which we seek solutions of the homogenous transport equation subject to

$$\Psi(x,\mu) \rightarrow \Psi_{0-}(x,\mu) \ , \qquad x \rightarrow \infty \tag{1.15}$$

where $\Psi(x,\mu)$ is the solution of the problem, and to the condition at $x = 0$,

$$\Psi(0,\mu) = 0 \qquad , \mu > 0 \tag{1.16}$$

The solution to this problem can be taken to be a linear combination of the fundamental solutions which vanish at infinity ~~plus~~ with Ψ_{0-}. Hence,

$$\Psi(x,\mu) = \Psi_{0-}(x,\mu) + a_{0+}\Psi_{0+}(x,\mu) + \int_0^1 A(\nu)\Psi_\nu(x,\mu)d\nu \tag{1.17}$$

The condition (1.16) then gives

$$-\varphi_{0-}(\mu) = a_{0+}\,\varphi_{0+}(\mu) + \int_0^1 A(\nu)\varphi_\nu(\mu)d\nu \ , \ \mu \geq 0 \tag{1.18}$$

By applying the orthogonality relations [1], a_{0+} and $A(\nu)$ can be found immediately:

$$\frac{W(\nu)N(\nu)}{\nu} A(\nu) = -\int_0^1 W(\mu)\ \varphi_\nu(\mu)\ \varphi_{0-}(\mu)d\mu \tag{1.19a}$$

or

$$A(\nu) = \frac{-c\nu_0\ X(-\nu_0)\ \varphi_{0-}(\nu)\nu^2}{(\nu_0-\nu)\gamma(\nu)N(\nu)} \tag{1.19b}$$

where

$$N(\nu) = \nu[(1-c\nu \tan h^{-1}\nu)^2 + c^2\pi^2\nu^2/4] \tag{1.19c}$$

Similarly

$$-[\tfrac{1}{2} c\nu_0]^2 \; X(\nu_0) \; a_{0+} = -\int_0^1 \gamma(\mu)(\nu_0-\mu) \; \varphi_{0+}(\mu) \; \varphi_{0-}(\mu)d\mu \tag{1.20a}$$

so that

$$a_{0+} = X(-\nu_0)/X(\nu_0) \tag{1.20b}$$

In the Albedo problem [1], in which we wish to obtain a solution to the homogeneous transport equation for $0 \leq x \leq \varphi$ subject to the two boundary conditions

$$\Psi_a(0,\mu) = \delta(\mu-\mu_0) \qquad \mu_0,\mu > 0 \tag{1.21a}$$

and

$$\lim_{x \to \infty} \Psi_a(x,\mu) = 0, \tag{1.21b}$$

the solution is taken in the form

$$\Psi_a(x,\mu) = a_{0+}\Psi_{0+}(x,\mu) + \int_0^1 A(\nu)\Psi_\nu(x,\mu)d\nu \tag{1.22}$$

where, the coefficients a_{0+} and $A(\nu)$ are obtained from the boundary conditions, Eq. (1.21a) using the half-range orthogonality formulas [1].

$$A(\nu) = \frac{\nu W(\mu_0) \; \varphi_\nu(\mu_0)}{N(\nu)W(\nu)} \tag{1.23}$$

and

$$a_{0+} = -2\gamma(\mu_0)/c\nu_0 \; X(\nu_0) \tag{1.24}$$

Thus from the two examples mentioned above, we see that the nature of $A(\nu)$ is ~~are~~ quite complicated and the integra~~tion~~ls involving this in Eqs. (1.17) and (1.22) cannot be evaluated analytically. Therefore it becomes necessary to find an approximation for the integral in the solution of the one-speed transport equation. This is the principal objective of the present thesis.

In the conventional P_N approximation, the exact solution $\Psi(x,\mu)$ given by Eq. (1.2a) is approximated in the form of a truncated Legendre polynomial expansion of the functions $\varphi_{0\pm}(\mu)$ and $\varphi_\nu(\mu)$ and this forms the basis of the P_N approximation [2]. This approximation is equivalent to

$$g_{N+1}(\nu) = 0, \tag{1.25}$$

where $g_n(\nu)$ are the coefficients in the expansion of $\Psi(x,\mu)$ as a sum and are given by

$$g_n(\nu) = \int_{-1}^{1} \varphi_\nu(\mu)\, P_n(\mu)\, d\mu \tag{1.25a}$$

Using Eq. (1.3) and (1.4) for $\varphi_\nu(\mu)$, we have

$$g_n(\mu) = c\nu Q_n(\nu) \quad , \quad \nu \notin (-1,1) \tag{1.26}$$

and

$$g_n(\nu) = c\nu P Q_n(\nu) + \lambda(\nu) P_n(\nu), \; \nu \in (-1,1) \tag{1.27}$$

where $Q_n(\nu)$ are Legendre polynomials of second kind [2] .

The Eq. (1.25) discretizes ν to (N+1) values. The conventional P_N approximation then considers the largest two of them as the asymptotic relaxation lengths and the remaining (N-1) as the transient ones. The two largest roots are obtained from Eq. (1.26) by imposing the condition that $g_o(\nu) = 1$.

In the transport-theoretic $P_N(TP_N)$ approximation, the infinite sums for the two asymptotic roots, i.e., $\varphi_{o\pm}(\mu)$ are retained and the series for the transient part is terminated. The solution is written as [2]

$$\Psi(x,\mu) = \bar{a}_{o+} \exp(-x/\nu_o)\, \varphi_{o+}(\mu) + \bar{a}_{o-} \exp(x/\nu_o)\, \varphi_{o-}(\mu)$$

$$+ \sum_{j=\text{roots}} A(\nu_j)\exp(-x/\nu_j) \sum_{n=0}^{N} \frac{2n+1}{2}\, g_n(\nu_j) P_n(\mu) \tag{1.28}$$

The coefficients in Eq. (1.28) are to be determined from the boundary conditions. Here the standard free surface Marshak boundary condition is modified to get the exact extrapolated end point. The solution of the source free Milne problem [2] is considered and the integral in the solution is replaced by a sum and the coefficients are obtained from the half-range orthogonality relations [2].

This TP_N formalism has been applied to the following problems [2]

1. The emergent angular distribution and leakage for the Milne problem
2. The scalar flux in Milne problem when $c = 1$
3. leakage in constant source Milne problem
4. the critical problem
5. Interface and source problems

and the results are compared with the exact values. The TP_N approximation gives significant improvements over conventional P_N method, except in those cases where the effect of the transients is especially important.

Another method known as the F_N method in Neutron-Transport theory [6] is used to solve the half-space albedo problem and the half-space constant-source Milne problem. The half-space problem is defined, for $c < 1$, by

$$\mu \frac{\partial \Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \Psi(x,\mu')d\mu' + a \tag{1.29}$$

$$\Psi(0,\mu) = 1-a \qquad \mu > 0 \tag{1.30}$$

and

$$\Psi(x,\mu) \to \frac{a}{1-c}, \text{ as } x \to \infty \tag{1.31}$$

Here $a = 1$ yields the usual constant-source Milne problem, and $a=0$ yields the half-space albedo problem. A particular solution of the problem is [7]

$$\Psi(x,\mu) = A(\nu_0)\varphi(\nu_0,\mu)\exp(-x/\nu_0) + \int_0^1 A(\nu)\varphi(\nu,\mu)\exp(-x/\nu)d\nu + \frac{a}{1-c}, \quad \mu > 0 \tag{1.32}$$

Then, on noting that

$$\int_{-1}^{1} [\Psi(x,\mu) - \frac{a}{1-c}]\, \mu\, \varphi(-\xi,\mu)d\mu = 0, \quad \xi \in P \tag{1.33}$$

where $\xi \in P \implies \xi = \nu_0$ or $\xi = \nu \in (0,1)$, we have

$$\frac{2}{c\xi} \int_0^1 \varphi(\xi,\mu)\Psi(0,-\mu)\mu\, d\mu = \frac{2a}{c} + (1-a)[1-\xi \log(1+ \frac{1}{\xi})] \qquad \xi \in P \tag{1.34}$$

The last equation has been obtained by splitting the interval $(-1,1)$ to $(-1,0)$ and $(0,1)$ and using the Eq. (1.30) in the last integral, i.e., for the interval $(0,1)$.

We enter for the emergent distribution,

$$\Psi(0,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \quad \mu > 0 \tag{1.35}$$

into Eq. (1.34) evaluated at selected values of ξ to obtain the F_N equations

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi_\beta) = \frac{2a}{c} + (1-a)[1-\xi_\beta \log(1+ \frac{1}{\xi_\beta})], \qquad \beta = 0,1,2,\ldots,N \tag{1.36}$$

where

$$B_\alpha(\xi) = \xi\, \beta_{\alpha-1}(\xi) - \frac{1}{\alpha+1}\,,\ \alpha \geq 1 \qquad (1.36a)$$

with

$$B_o(\xi) = \frac{2}{c} - 1 - \xi \log(1+\frac{1}{\xi}) = \nu_o \log \frac{\nu_o}{\nu_o - 1} - 1 \qquad (1.36b)$$

Then $\xi_o = \nu_o$, $\xi_1 = 0$, $\xi_2 = 1$, and the remaining ξ_β are spaced equally in the interval [0,1] and is one possible set of collocation points that has been used in the literature [6].

The leakage given by

$$J = \int_{-1}^{1} \mu \Psi(0,+\mu)d\mu$$

$$= \int_{-1}^{0} \mu\Psi(0,\mu)d\mu + \int_{0}^{1} \mu\Psi(0,\mu)d\mu$$

$$= \int_{0}^{1} \mu\Psi(0,-\mu)d\mu \qquad (1.37)$$

is calculated for the cases $\alpha = 0$ and $a = 1$ and compared with the exact values. This method, although particularly concise, yields excellent numerical results for the problems considered.

In the present thesis, we utilise the concept of a rational function approximation introduced by Sengupta [3], [4] for the singular eigen-distribution to reduce the transient integrals of Eq. (1.2b) to a sum. The Legendre or Chebyshev polynomials are usually used as the complete

set and their zeros as collocation points for obtaining the approximate solutions of Neutron Transport equation. Since the actual set consists of the case eigen-functions w.r.t. the weight function $W(\mu)$ it is not the best possible to use the classical orthogonal polynomials in transport theory. Therefore it becomes necessary to orthogonalise the fundamental set $\{\mu^i\}_{i=0}^{\infty}$ w.r.t. $W(\mu)$ and use this as a proper substitute for the case eigen-functions $\varphi_{0\pm}(\mu)$ and $\varphi_\nu(\mu)$ [5].

In our present work, a weight function $\bar{W}(\mu)$ is used in the calculations instead of $W(\mu)$ given by Eq. (1.8) to obtain the coefficients occuring in the solution of the transport equation and this has been properly justified in the next chapter. A new set of orthogonal polynomials $C_i(\mu)$ w.r.t. the weight function $\bar{W}(\mu)$ has been constructed Ref.[5] and the values of μ and ν are taken as the zeros of these polynomials depending on the number of transient terms considered. Then the coefficients are obtained for the problems given below

(i) standard Milne problem

(ii) constant-source Milne problem

(iii) half-space Albedo problem

and then the emergent angular distribution and leakage are calculated and compared with the exact values. This method

CHAPTER 2

METHOD FORMULATION

1. TRANSPORT THEORETIC APPROXIMATION [3], [4], [5], [8]

The solution of the one-speed equation [3] is given by Eqs. (1.2a) and (1.2b) and the Case eigen-functions are given by Eqs. (1.3) and (1.4). The constants $a_{0\pm}$ and $A(\nu)$ occuring in the solution are determined by using the orthogonality integrals [1] given by Eqs. (1.6) and (1.7) for the full-and half-range problems. Here X-function is given by Eq. (1.9) and W-function by Eq. (1.8).

In this approximation, it is reasoned as follows [3]

(i) The discretization of $\nu \varepsilon (-1,1)$ or $(0,1)$ must be done such that these roots, $\nu_j, j=1,2...$ are consistent with the particular choice of ν_o. The P_N approximation has this desirable property, as all its $(N+1)$ ν_j are solutions of a polynomial equation of degree $N+1$, Eq. (1.25). In contrast, by requiring the asymptotic ν_o to satisfy Eq. (1.5) and the transient ν_j the equation (1.25), the TP_N procedure [2] violates this property.

(ii) The natural basis functions for the solution of the transport equation are the case eigen-functions $\varphi_{o+}(\mu)$, $\varphi_{o-}(\mu)$, $\varphi_{\nu}(\mu)$. Any other basis such as the $\{P_n(\mu)\}_{n=0}^{N}$

(as in the P_N case), or a combination $\{\varphi_{o+}(\mu), \varphi_{o-}(\mu), \{P_n(\mu)\}\}$ (as in the TP_N case), is likely to be unsatisfactory.

(iii) By the very nature of the eigen-functions, a rational function approximation is expected to be superior to a standard polynomial approximation.

In the case of a half-range problem, a ~~single~~ simple but remarkably accurate expression for $W(\mu)$, [3]

$$\bar{W}(\mu) = \frac{c}{2\Omega(1-c)} \quad \frac{\mu\nu_o\sqrt{1-c} + \mu^2}{\nu_o + \mu} \tag{2.1}$$

where Ω is taken to be a constant has been constructed [3] for use in the orthogonality integrals (1.7). The value of Ω is obtained by evaluating the orthogonality integral (1.7c). Also with this constant Ω, a first iterant of $\Omega(-\mu)$ from Eq. (1.10)-and using this, $X(-\mu)$ from Eq. (1.9)-are obtained [3].

Now, the solution of the transport equation (1.1) can be written as

$$\Psi(x,\mu) = a_{o+}\exp(-x/\nu_o)\,\varphi_{o+}(\mu) + a_{o-}\exp(a/\nu_o)\varphi_{o-}(\mu)$$
$$+ \sum_j A(\nu_j)\exp(-x/\nu_j)\,\varphi_\varepsilon(\nu_j,\mu) \tag{2.2}$$

where $\varphi_\varepsilon(\nu,\mu)$ is the proper rational function approximation to $\varphi(\nu,\mu)$, $\nu \in (-1,1)$ and is given by [4]

$$\varphi_{\varepsilon}(\nu,\mu) = \frac{c\nu}{2} \frac{(\nu-\mu)}{(\nu-\mu)^2+\varepsilon^2} + \frac{\lambda\varepsilon(\nu)}{\pi_{\varepsilon}} \frac{\varepsilon}{[(\nu-\mu)^2+\varepsilon^2]},$$

$$\nu \in (-1,1) \tag{2.3}$$

The constants in Eq. (2.2) can be determined using the boundary conditions and orthogonality relations satisfied by the set $\{\varphi_{0\pm}(\mu), \varphi_{\varepsilon}(\nu_j,\mu)\}$. The function $\varphi_{\varepsilon}(\nu,\mu)$, as $\varepsilon \to 0$, tends to the sum of the two distributions that comprise $\varphi(\nu,\mu)$. One is the Principal value distribution and the other is the delta distribution.

Using the normalisation of φ_{ε}, we get $\lambda\varepsilon(\nu)$ from Eq. (2.3),

$$\lambda\varepsilon(\nu) = \frac{\pi_{\varepsilon}}{\tan^{-1}\frac{1+\nu}{\varepsilon} + \tan^{-1}\frac{1-\nu}{\varepsilon}} \left[1 - \frac{c\nu}{4} \ln \frac{(1+\nu)^2+\varepsilon^2}{(1-\nu)^2+\varepsilon^2}\right] \tag{2.4}$$

where

$$\pi\varepsilon = 2 \tan^{-1} 1/\varepsilon \tag{2.5}$$

and $\lambda\varepsilon(\nu) \to \lambda(\nu)$ as $\varepsilon \to 0$.

The convergence of the two distributions that comprise (ν,μ) has been verified [8] by finding the solution of Love's Integral equation

$$U_{\varepsilon}(s) = 1 - \frac{\varepsilon}{\pi} \int_{-1}^{1} \frac{U_{\varepsilon}(t)}{(s-t)^2+\varepsilon^2} dt \tag{2.6}$$

and the Cauchy Principal integral

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(t)}{t-x} dt = P_0 , \tag{2.7}$$

where P_0 is a constant.

In our present work, the solution of the one-speed equation (1.1) is taken in the form given by Eq. (2.2). Then the given boundary condition is made use of and the coefficients a_{0+}, a_{0-} and $\{A(\nu_j)\}$; are determined by considering the μ_j's as the roots of $C_{N+1}(\mu_j) = 0$ and ν_j's as the roots of $C_N(\nu_j) = 0$ where $C_N(\mu)$ are the orthogonal polynomials constructed in [5] and details of which are given in next section. The value of ε is determined from the relation given by Eq.

$$\varepsilon = \frac{1}{2N \tan^{-1} 1/\varepsilon} \tag{2.7a}$$

depending on the number of transient terms used. Once the coefficients are determined then the emergent angular distribution and Leakage can be calculated for various problems considered.

2. JUSTIFICATION OF WEIGHT FUNCTION

A weight function $\bar{W}(\mu)$ given by Eq. (2.1) has been constructed in [3]. Considering the simplicity of $\bar{W}(\mu)$ as compared to the exact $W(\mu)$ given by Eq. (1.8), the accuracy of the former is remarkable. With this $\bar{W}(\mu)$, the orthogonality integral (1.7c) are evaluated to give

$$\frac{c}{2\Omega(1-c)} \left(\frac{1}{2}\, c\nu_o\right)^2 (\alpha C_1 + C_2) = -\left(\frac{1}{2}\, c\nu_o\right)^2 X(\nu_o) \tag{2.8}$$

and

$$\frac{c}{2\Omega(1-c)} \left(\frac{1}{2}\, c\nu_o\right)^2 (\alpha D_1 + D_2) = \left(\frac{1}{2}\, c\nu_o\right)^2 X(-\nu_o) \tag{2.9}$$

where,

$$C_1 = \frac{1}{2\nu_o(\nu_o - 1)} - \frac{1}{2c\nu_o^{\,2}} ;$$

$$C_2 = \frac{1}{2(\nu_o - 1)} + \frac{1}{2c\nu_o} - \ln \frac{\nu_o}{\nu_o - 1}$$

$$D_1 = \frac{1}{2c\nu_o^{\alpha}} - \frac{1}{2\nu_o(\nu_o + 1)} ;$$

$$D_2 = \frac{1}{2(\nu_o + 1)} - \frac{3}{2c\nu_o} + \ln \frac{\nu_o}{\nu_o - 1}$$

The two equations (2.8) and (2.9) are solved to get Ω_+ and Ω_-. If Ω is very nearly constant, which is assumed to be the case in Eq. (2.1), then the solution of the above two equations for Ω, i.e. Ω_+ and Ω_-, will be nearly the same. To take care of the weak dependence of Ω on μ to a satisfactory degree, we use the average of Ω_+ and Ω_- i.e. let

$$\Omega = \frac{1}{2}\left(\Omega_+ + \Omega_-\right) \tag{2.10}$$

As an independent check on the utility of this $\overline{W}(\mu)$ in evaluating integrals of the type (1.7), the integral below was evaluated [3] as shown in Eq. (2.11),

$$\int_0^1 \bar{W}(\mu)\,\varphi_{0+}(\mu)d\mu = (c^2\nu_0)\,[4\Omega(1-c)]^{-1}(\alpha F_1+F_\alpha) \qquad (2.11)$$

where

$$F_n = \int_0^1 \frac{\mu^n}{\nu_0^2-\mu^2}\,d\mu$$

$$= -\left(\frac{1}{n-1}+\frac{\nu_0^2}{n-3}+\frac{\nu_0^4}{n-5}+\ldots\right)+\begin{cases}\nu_0^{n-2}/c & n \text{ even}\\ \nu_0 \ln[\nu_0/(\nu_0^2-1)^{1/2}] & n \text{ odd}\end{cases}$$

and compared it with the exact value $\frac{1}{2}c\nu_0$. The ratio of the approximate to exact integrals for different c are 0.999594, 0.999530, 0.999693 and 0.999832 for $c = 0.2, 0.4, 0.6, 0.8$ and 0.9 respectively.

As a further check, we have evaluated the integral I given by

$$I = \int_0^1 \bar{W}(\mu)\,\varphi_{0+}(\mu)\,\varphi(\nu,\mu)d\mu \qquad (2.11a)$$

and compared it with its exact value (for the real $W(\mu)$) of zero

$$I = \frac{c^3\nu\nu_0}{8\Omega(1-c)}I_1 + \frac{c^2\nu_0}{4\Omega(1-c)}\,\frac{[\nu\nu_0\sqrt{1-c}+\nu^2]}{(\nu_0^2-\nu^2)}\,\lambda(\nu) \qquad (2.11b)$$

where

$$I_1 = \frac{A'}{2}\ln\frac{\nu_0^2}{\nu_0^2-1} + B'\,\frac{1}{2\nu_0}\ln\frac{\nu_0+1}{\nu_0-1} + c'\ln\frac{\nu}{1-\nu} \qquad (2.11c)$$

$$A' = -\frac{\nu_o[\nu\sqrt{1-c} + \nu_o]}{(\nu_o^2 - \nu^2)}$$

$$B' = -\frac{\nu_o^2[\nu + \nu_o\sqrt{1-c}]}{(\nu_o^2 - \nu^2)}$$

$$C' = \frac{\nu[\nu + \nu_o\sqrt{1-c}\,]}{(\nu_o^2 - \nu^2)}$$

The value of I has been obtained for different values of C and ν and the outputs are attached in APPENDIX-A. Considering the accuracy of these two examples, we can say that the simple analytic approximation (2.1) to the transcendental weight function $W(\mu)$, Eqs. (1.8)-(1.10), can be used with confidence.

A first iterant of $X(-\mu)$ can be obtained from Eq. (1.9) with $\Omega(-\mu)$ replaced by the constant Ω, i.e.,

$$X_o(-\mu) = \frac{1}{\alpha+\mu}\,\Omega \tag{2.12}$$

A first iterant of $\Omega(-\mu)$ is obtained from Eq. (1.10) as

$$\Omega_1(-\mu) = [1 - \frac{c)_o^2}{2}\,\frac{\mu}{\Omega}\,W] \tag{2.13}$$

where

$$W = \alpha_1 \ln\frac{1+\nu_o}{\nu_o} + \alpha_2 \ln\frac{\nu_o - 1}{\nu_o} + \alpha_3 \ln\frac{1+\mu}{\mu}$$

$$\alpha_1 = \frac{c}{2\nu_o(\nu_o-\mu)(1-c)}$$

$$\alpha_2 = \frac{c}{2\nu_o(\nu_o+\mu)(1-c)}$$

$$\alpha_3 = X^2(0) - \frac{c}{(1-c)(\nu_o^2-\mu^2)}$$

with this $\Omega_1(-\mu)$, a second iterant of $X(-\mu)$ is obtained as

$$X_1(-\mu) = \frac{1}{(\alpha+\mu)}\,\Omega(-\mu) \tag{2.14}$$

The values of $X_o(-\mu)$ given by Eq. (2.12), $X_1(-\mu)$ given by Eq. (2.15) and the ERROR given by

$$\text{ERROR} = \frac{(\text{EXACT VALUE} - \text{CALCULATED VALUE})}{\text{EXACT VALUE}} \times 100 \tag{2.15}$$

are calculated for various values of μ and the outputs are attached in APPENDIX B. We observe from the results that the error are very less and the above, therefore, constitutes an extremely reliable approximation to Case's W and X functions.

3. ORTHOGONAL POLYNOMIALS [5]

$\{1,\mu,\mu^2\}$ is orthogonalised w.r.t. $\overline{W}(\mu)$ given by Eq. (2.4), $0 \le \mu \le 1$, using the prescription given by Golub and Welseh.[9]. Let

$$M_{ij} = \int_0^1 \overline{W}(\mu)\,\mu^{i+j-2}d\mu, \quad i,j=1,2,\ldots,N+1 \tag{2.16}$$

and

$$\nu_{ij} = \frac{1}{\nu_{ii}} (M_{ij} - \sum_{k=1}^{i-1} \nu_{ki} \nu_{kj}), \quad i < j \tag{2.17}$$

where

$$\nu_{ii} = (M_{ii} - \sum_{k=1}^{i-1} \nu_{ki}^{2})^{1/2} \tag{2.18}$$

Now construct

$$\alpha_i = \frac{\nu_{i,i+1}}{\nu_{i,i}} - \frac{\nu_{i-1,i}}{\nu_{i-1,i-1}} ;$$

$$\beta_i = \frac{\nu_{i+1,i+1}}{\nu_{i,i}} , \quad i=1,2,\ldots,N \tag{2.19}$$

where $\nu_{o,o} = 1$, $\nu_{o,1} = 0$.

Then with $C_o=1$, $C_{-1} = 0$, the set $\{C_i(\mu)\}_{i=0}^{N}$ obtained from the recurrence relation

$$\beta_i C_i(\mu) = (\mu-\alpha_i) C_{i-1}(\mu) - \beta_{i-1}C_{i-2}(\mu), \quad i=1,2,\ldots,N \tag{2.20}$$

form a complete orthonormal system in (0,1) w.r.t., $\bar{W}(\mu)$.

For our present problem,

$$M_{ij} = \frac{c}{2\Omega(1-c)} \int_0^1 \frac{\alpha+\mu}{\nu_o+\mu} \mu^{i+j-1} d\mu \tag{2.21}$$

which is expressed in terms of

$$I_n = \int_0^1 \frac{\mu^n}{\nu_o+\mu} d\mu = \frac{1}{n} [1-\nu_o n I_{n-1}] \tag{2.22}$$

where

$$I_o = \ln \frac{\nu_o + 1}{\nu_o}, \tag{2.23}$$

as

$$M_{ij} = \frac{c}{2\Omega(1-c)} (\alpha\, I_{i+j-1} + I_{i-j}) \tag{2.24}$$

The value of M_{ij} can be evaluated recursively, and hence ν_{ij}, α_i, β_i and finally $\{C_i(\mu)\}_0^N$ is obtained from Eq. (2.20). The orthogonal polynomials $C_i(\mu)$ $i=0,1,\ldots,5$ has been developed and presented below.

By Eqs. (2.18), (2.19), (2.20) and (2.21), we have

$$\alpha_1 = \frac{r_{1,2}}{r_{1,1}}; \quad \beta_1 = \frac{r_{2,2}}{r_{1,1}}$$

$$r_{1,1} = (M_{11})^{1/2} = \delta_o^{1/2}$$

where

$$\delta_n = R\ I_{n+1} + S\ I_{n+2}$$

$$R = \frac{c\nu_o \sqrt{1-c}}{2\Omega(1-c)}; \quad S = \frac{c}{2\Omega(1-c)}$$

and I_n's are given by Eq. (2.23) and (2.24).

$$r_{1,2} = \frac{M_{12}}{r_{1,1}} = \frac{\delta_1}{\delta_o^{1/2}}$$

$$r_{2,2} = (M_{22} - r_{1,2}^2)^{1/2}$$

$$= (\delta_2 - \frac{\delta_1^2}{\delta_o})^{1/2}$$

$$\alpha_1 = \frac{\delta_1}{\delta_o} \; ; \; \beta_1 = \frac{1}{\delta_o} (\delta_o \delta_2 - \delta_1^2)^{1/2}$$

$$C_1(\mu) = (\mu - \alpha_1)/\beta_1$$

Also

$$\alpha_2 = \frac{r_{2,3}}{r_{2,2}} - \frac{r_{1,2}}{r_{1,1}}$$

$$\beta_2 = \frac{r_{3,3}}{r_{2,2}}$$

$$r_{2,3} = (M_{23} - r_{1,2}\, r_{1,3})/r_{2,2}$$

$$= (\delta_3 - r_{1,2}\, r_{1,3})/r_{2,2}$$

$$r_{3,3} = (M_{33} - r_{1,3}^2 - r_{2,3}^2)^{1/2}$$

$$= (\delta_4 - r_{1,3}^2 - r_{2,3}^2)^{1/2}$$

$$C_2(\mu) = [(\mu - \alpha_2)\, C_1(\mu) - \beta_1]/\beta_2$$

$$= [(\mu - \alpha_2)(\mu - \alpha_1) - \beta_1^2]/\beta_1 \beta_2$$

Similarly, we have

$$\alpha_3 = \frac{r_{3,4}}{r_{3,3}} - \frac{r_{2,3}}{r_{2,2}}$$

$$\beta_3 = \frac{r_{4,4}}{r_{3,3}}$$

$$r_{3,4} = (M_{34} - r_{1,3}\, r_{1,4} - r_{2,3}\, r_{2,4})/r_{3,3}$$

$$r_{1,4} = M_{14}/r_{1,1} = \delta_3/\delta_o^{1/2}$$

$$r_{2,4} = (M_{24} - r_{1,2}\, r_{1,4})^{1/2}$$

$$= (\delta_4 - r_{1,2}\, r_{1,4})^{1/2}$$

$$r_{4,4} = (M_{44} - r_{1,4}^2 - r_{2,4}^2 - r_{3,4}^2)^{1/2}$$

$$= (\delta_6 - r_{1,4}^2 - r_{2,4}^2 - r_{3,4}^2)^{1/2}$$

$$c_3(\mu) = [(\mu - \alpha_3)\, c_2(\mu) - \beta_2\, c_1(\mu)]/\beta_3 \ .$$

$$\alpha_4 = \frac{r_{4,5}}{r_{4,4}} - \frac{r_{3,4}}{r_{3,3}}$$

$$\beta_4 = \frac{r_{5,5}}{r_{4,4}}$$

$$r_{1,5} = M_{15}/r_{1,1} = \delta_4/\delta_o^{1/2}$$

$$r_{2,5} = (M_{25} - r_{1,2}\, r_{1,5})/r_{2,2}$$

$$= (\delta_6 - r_{1,2}\, r_{1,5})/r_{2,2}$$

$$r_{3,5} = (M_{35} - r_{1,3}\, r_{1,5} - r_{2,3}\, r_{2,5})/r_{3,3}$$

$$= (\delta_6 - r_{1,3}\, r_{1,5} - r_{2,3}\, r_{2,5})/r_{3,3}$$

$$r_{4,5} = (M_{45} - r_{1,4}\, r_{1,5} - r_{2,4}\, r_{2,5} - r_{3,4}\, r_{3,5})/r_{4,4}$$

where

$$M_{45} = \delta_7$$

$$r_{5,5} = (M_{55} - r_{1,5}^2 - r_{2,5}^2 - r_{3,5}^2 - r_{4,5}^2)^{1/2}$$

$$M_{55} = \delta_8$$

This gives

$$c_4(\mu) = [(\mu-\alpha_4)\, c_3(\mu) - \beta_3\, c_2(\mu)]/\beta_4 \, .$$

Finally

$$\alpha_5 = \frac{r_{5,6}}{r_{5,5}} - \frac{r_{4,5}}{r_{4,4}} \; ; \; \beta_5 = \frac{r_{6,6}}{r_{5,5}}$$

$$r_{1,6} = M_{16}/r_{1,1} = \delta_5/\delta_o^{1/2}$$

$$r_{2,6} = (M_{26} - r_{1,2}\, r_{1,6})/r_{2,2}$$

$$M_{26} = \delta_6$$

$$r_{3,6} = (M_{36} - r_{1,3}\, r_{1,6} - r_{2,3}\, r_{2,6})/r_{3,3}$$

$$M_{36} = \delta_7$$

$$r_{4,6} = (M_{46} - r_{1,4}\, r_{1,6} - r_{2,4}\, r_{2,6} - r_{3,4}\, r_{3,6})/r_{4,4}$$

$$M_{46} = \delta_8$$

$$r_{5,6} = (M_{56} - r_{1,5}\, r_{1,6} - r_{2,5}\, r_{2,6} - r_{3,5} r_{3,6} - r_{4,5} r_{4,6})/r_{5,5}$$

$$M_{56} = \delta_9$$

$$r_{6,6} = (M_{66} - r_{1,6}^2 - r_{2,6}^2 - r_{3,6}^2 - r_{4,6}^2 - r_{5,6}^2)^{1/2},$$

$$M_{66} = \delta_{10}$$

$$C_5(\mu) = [(\mu-\alpha_5)\ C_5(\mu) - \beta_4\ C_3(\mu)]/\beta_5 \ .$$

Note that

$$C_o(\mu) = 1.$$

The roots fof $C_i(\mu) = 0$ for N=1 to 5 have also been obtained by bisection method and the program and the outputs are attached in APPENDIX-C.

4. APPLICATIONS

The method discussed in section 1 of this chapter has been applied to the following problems

(i) Standard source-free Milne Problem [1]

(ii) Constant-source Milne Problem [1]

(iii) Half-space Albedo Problem [1]

and they are discussed below.

(i) Standard source-free Milne Problem

The solution of the standard source-free Milne Problem is given by

$$\Psi(x,\mu) = a_{0+} \exp(-x/\mu_0) \, \varphi_{0+}(\mu) + \exp(x/\nu_0) \, \varphi_{0-}(\mu)$$

$$+ \sum_{j=1}^{N} A(\nu_j) \exp(-x/\nu_j) \, \varphi_\varepsilon(\nu_j,\mu) \tag{2.25}$$

with boundary condition at $x = 0$,

$$\Psi(0,\mu) = 0 \quad , \qquad \mu > 0 \tag{2.26}$$

Note that in Eq. (2.25), the value of a_{0-} is taken to be unity and $\varphi_\varepsilon(\nu,\mu)$ is given by Eq. (2.3).

Then applying Eq. (2.26) to Eq. (2.25), we get

$$a_{0+} \varphi_{0+}(\mu) + \varphi_{0-}(\mu) + \sum_j A(\nu_j) \, \varphi_\varepsilon(\nu_j,\mu) = 0 \tag{2.27}$$

The coefficients a_{0+} and $\{A(\nu_j)\}_{j=1}^{N}$ can be obtained by taking μ's as the roots of $C_{N+1}(\mu = 0$, ν's as the roots of $C_N(\nu) = 0$ and the value of ε from Eq. (2.7a) for N equal to the number of transient terms considered.

The emergent angular distribution $\Psi(0,\mu)$ is given by

$$\Psi(0,\mu) = a_{0+} \varphi_{0+}(\mu) + \varphi_{0-}(\mu) + \sum_j A(\nu_j) \varphi_{\nu j}(\mu),$$

$$\mu > 0, \quad \nu > 0 \tag{2.28}$$

Note that for $\mu < 0$, $\varphi_\varepsilon(\nu_j,\mu) \to \varphi_{\nu j}(\mu)$ as there is no chance of the denominator of

$$\varphi_\nu(-\mu) = \frac{c\nu}{2(\nu+\mu)}$$

becoming zero. Hence the case of Principal value and Delta distribution doesn't arise.

The leakage J is given by

$$J = \int_{-1}^{0} \mu \Psi(0,\mu) d\mu$$

$$= a_{0+} \int_{-1}^{0} \mu \varphi_{0+}(\mu) d\mu + \int_{-1}^{0} \mu \varphi_{0-}(\mu) d\mu + \sum_j A(\nu_j) \int_{-1}^{0} \mu \varphi_\varepsilon(\nu_j,\mu) d\mu$$

$$= -a_{0+}\left(\frac{c\nu_0}{2}\right)\left[1-\nu_0 \ln \frac{\nu_0+1}{\nu_0}\right]+\left(\frac{c\nu_0}{2}\right)\left[1-\nu_0 \ln \frac{\nu_0}{\nu_0-1}\right]$$

$$- \sum_j A(\nu_j) \frac{c\nu_j}{2} \left[1-\nu_j \ln \frac{\nu_j+1}{\nu_j}\right] \tag{2.29}$$

Note that in the last integral of the expression J, $\varphi_\varepsilon(\nu_j,\mu) \to \varphi_{\nu j}(\mu)$ as $\mu < 0$.

(ii) <u>Constant-source Milne problem</u>

a) The solution of Eq. (1.29) with a=1 and boundary condition given by Eq. (1.30) and (1.31) is

$$\Psi(x,\mu) = a_{0+} \varphi_{0+}(\mu) \exp(-x/\nu_0) + \sum_j A(\nu_j)\varphi_\varepsilon(\nu_j,\mu)\exp(-x/\nu_j)$$
$$+ \frac{1}{1-c} \tag{2.30}$$

Using the boundary conditions, the coefficients are obtained from the equation

$$a_{0+} \varphi_{0+}(\mu) + \sum_j A(\nu_j) \varphi_\varepsilon(\nu_j,\mu) + \frac{1}{1-c} = 0 \tag{2.31}$$

as in the standard source-free Milne problem. After

determining the coefficients a_{0+} and $\{A(\nu_j)\}$, the leakage is calculated by

$$J = \int_{-1}^{0} \mu\ \Psi(0,\mu)d\mu$$

$$= a_{0+} \int_{-1}^{0} \mu\ \varphi_{0+}(\mu)d\mu + \sum_j A(\nu_j) \int_{-1}^{0} \mu\varphi_\varepsilon(\nu_j,\mu)d\mu + \frac{1}{1-c} \int_{-1}^{0} \mu\ d\mu$$

$$= a_{0+} \int_{-1}^{0} \mu\varphi_{0+}(\mu)d\mu + \sum_j A(\nu_j) \int_{-1}^{0} \mu\varphi_{\nu j}(\mu)d\mu + \frac{1}{1-c} \int_{-1}^{0} \mu\ d\mu$$

$$= -a_{0+} \frac{c\nu_0}{2} [1-\nu_0 \ln \frac{\nu_0+1}{\nu_0}] - \sum_j A(\nu_j) \frac{c\nu_j}{2}[1-\nu_j \ln \frac{\nu_j+1}{\nu_j}] - \frac{1}{2(1-c)} \tag{2.32}$$

b) The solution of Eq. (1.29) with a=0 and boundary condition given by Eq. (1.30) and (1.31) is

$$\Psi(x,\mu) = a_{0+} \varphi_{0+}(\mu) \exp(-x/\nu_0) + \sum_j A(\nu_j)\varphi_\varepsilon(\nu_j,\mu)\exp(-x/\nu_j) \tag{2.33}$$

using the boundary condition, the coefficients are determined from

$$a_{0+} \varphi_{0+}(\mu) + \sum_j A(\nu_j)\ \varphi_\varepsilon(\nu_j,\mu) = 0 \tag{2.34}$$

as before. Then the leakage is obtained from

$$J = \int_{-1}^{0} \mu\Psi(0,\mu)d\mu$$

$$= a_{0+} \int_{-1}^{0} \mu\varphi_{0+}(\mu)d\mu + \sum_j A(\nu_j) \int_{-1}^{0} \mu\varphi_\varepsilon(\nu_j,\mu)d\mu$$

5. RESULTS AND CONCLUSION

The values of ν_o, Ω and Z_o for various values of C ranging from 0.1(0.1) 0.9 are presented in Table 1. The value of Ω is calculated from the Eq. (2.10).

The emergent angular distribution and the leakage for the standard source-free Milne problem are calculated using Eq. (2.28) and (2.29) and the percentage errors using Eq. (2.15) and are presented in Tables 2 and 3. The exact values are taken from [2].

The leakage for the constant source Milne problem with a=0 is calculated using Eq. (2.35) and the results with the percentage errors are presented in Table 4. The exact values are taken from [7].

The leakage for the constant-source Milne problem with a=1 is calculated using Eq. (2.32) and the results with the percentage errors are presented in Table 5. The exact values are taken from [7].

In all these cases, it has been observed that the calculated values of the emergent angular distribution and the leakage are very reliable. The calculated values converges to the exact values as the number of transients terms is increased. Thus the use of μ's and ν's as the zeros of the orthogonal polynomials given by Eq. (2.20) and ε from the Eq. (2.7a) is justified.

Further the results will improve if the orthogonality integrals (1.7) w.r.t. μ are used in obtaining the coefficients $a_{o\pm}$ and $\{A(\nu_j)\}$ and ν's as the zeros of the orthogonal polynomials given by Eq. (2.20) and ε from the Eq. (2.7a).

Table 2.1

Values of ν_o, Ω and z_o for different c's

c	Ω	ν_o	z_o
0.1	0.990542	1.0	8.539
0.2	0.981642	1.000091	3.9255
0.3	0.973524	1.002593	2.497
0.4	0.966199	1.014586	1.8263
0.5	0.959548	1.044382	1.4414
0.6	0.953451	1.102132	1.1925
0.7	0.947812	1.206804	1.0181
0.8	0.942561	1.407634	0.88913
0.9	0.937643	1.903205	0.7896

Table 2.2

Emergent Angular Distribution of Standard Milne Problem

ε		$\mu = -1$		$\mu = -0.8$		$\mu = -0.6$		$\mu = -0.4$		$\mu = -0.2$		$\mu = 0.0$	
		Ψ	PER	Ψ	PER	Ψ	PER	Ψ	PER	Ψ	PER	Ψ	PER
0.2	0.4289	1100.3	0.486	0.4983	0.521	0.2481	0.763	0.1643	1.08	0.1218	1.48	0.095	3.18
	0.1794	1100.3	0.486	0.4968	0.226	0.2464	0.093	0.1623	-0.11	0.1194	-0.52	0.092	-0.59
0.6	0.4289	3.2039	-0.08	1.0564	-0.37	0.6145	-0.82	0.4189	-1.58	0.3026	-2.79	0.217	-2.38
	0.1794	3.2045	-0.06	1.057	-0.30	0.6154	-0.69	0.4198	-1.36	0.303	-2.62	0.2147	-3.49
0.9	0.4289	0.8262	2.25	0.642	2.75	0.5075	3.09	0.4002	3.15	0.3063	2.41	0.2138	3.14
	0.1794	0.8109	0.36	0.6269	0.34	0.4929	0.16	0.3867	-0.32	0.2944	-1.56	0.2025	-2.33

Table 2.3

Leakage of the standard Milne problems

C	ε	LEAKAGE	EXACT	PER
0.2	0.4289	-0.8326	-0.8266	-0.721
	0.1794	-0.8293	-0.8266	-0.326
0.3	0.4289	-0.7499	-0.7427	-0.969
	0.1794	-0.7444	-0.7427	-0.234
0.4	0.4289	-0.6724	-0.6627	-1.457
	0.1794	-0.6641	-0.6627	-0.217
0.5	0.4289	-0.6016	-0.5881	-2.289
	0.1794	-0.5897	-0.5881	-0.272
0.6	0.4289	-0.5363	-0.5170	-3.727
	0.1794	-0.5194	-0.5170	-0.46
0.7	0.4289	-0.4733	-0.4466	-5.969
	0.1794	-0.4495	-0.4466	-0.643
0.8	0.4289	-0.4046	-0.3713	-9.456
	0.1794	-0.3741	-0.3713	-0.753
0.9	0.4289	-0.3199	-0.2772	-15.39
	0.1794	-0.2799	-0.2772	-0.957

Table 2.4

Leakage of the constant-source Milne problem with a=0

C	ε	2* J	EXACT	PER
0.1	0.4289	-0.0079	-0.0217	63.26
	0.1794	-0.0056	-0.0217	73.93
0.2	0.4289	-0.0539	-0.0463	-16.47
	0.1794	-0.0487	-0.0463	- 6.295
0.3	0.4289	-0.096	-0.0745	-29.27
	0.1794	-0.089	-0.0745	-20.61
0.4	0.4289	-0.1362	-0.1073	-26.95
	0.1794	-0.1288	-0.1073	-20.08
0.5	0.4289	-0.1760	-0.1465	-20.162
	0.1794	-0.1688	-0.1465	-15.213
0.6	0.4289	-0.2187	-0.1947	-12.31
	0.1794	-0.2136	-0.1947	- 9.699
0.7	0.4289	-0.2686	-0.2566	- 4.691
	0.1794	-0.2693	-0.2566	- 4.941
0.8	0.4289	-0.3357	-0.3419	+ 1.819
	0.1794	-0.3471	-0.3419	- 1.513
0.9	0.4289	-0.44919	-0.4780	+ 6.025
	0.1794	-0.4758	-0.4780	+ 0.466

Table 2.5

Leakage of the constant-source Milne problem with a=1

	ε	2* J	EXACT	PER
0.1	0.4289	-1.1023	-1.087	-1.403
	0.1794	-1.1004	-1.087	-1.23
0.2	0.4289	-1.1827	-1.192	+0.784
	0.1794	-1.1891	-1.192	+0.242
0.3	0.4289	-1.2911	-1.322	+2.339
	0.1794	-1.3004	-1.322	+1.634
0.4	0.4289	-1.4396	-1.4880	3.252
	0.1794	-1.4519	-1.4880	2.424
0.5	0.4289	-1.6479	-1.7070	3.465
	0.1794	-1.6624	-1.7070	2.611
0.6	0.4289	-1.9533	-2.013	2.964
	0.1794	-1.966	-2.013	2.333
0.7	0.4289	-2.438	-2.478	1.619
	0.1794	-2.436	-2.478	1.706
0.8	0.4289	-3.321	-3.291	-0.930
	0.1794	-3.2646	-3.291	0.801
0.9	0.4289	-5.508	-5.220	-5.518
	0.1794	-5.242	-5.220	-0.427

APPENDIX-A

Value of the Orthogonal Integral

$$\int_0^1 W(\mu)\ \phi_{0+}(\mu)\ \phi_{\nu}(\mu)\,d\mu$$

```
            C= .10
NEW=.10      VAL= 0.64571596E-06
NEW=.20      VAL= 0.72916532E-07
NEW=.30      VAL=-0.83637990E-06
NEW=.40      VAL=-0.18397796E-05
NEW=.50      VAL=-0.28434109E-05
NEW=.60      VAL=-0.38081095E-05
NEW=.70      VAL=-0.47182100E-05
NEW=.80      VAL=-0.55689063E-05
NEW=.90      VAL=-0.63605470E-05
            C= .20
NEW=.10      VAL= 0.53538503E-05
NEW=.20      VAL= 0.70729400E-06
NEW=.30      VAL=-0.66299639E-05
NEW=.40      VAL=-0.14690149E-04
NEW=.50      VAL=-0.22723483E-04
NEW=.60      VAL=-0.30422900E-04
NEW=.70      VAL=-0.37669183E-04
NEW=.80      VAL=-0.44428799E-04
NEW=.90      VAL=-0.50708265E-04
            C= .30
NEW=.10      VAL= 0.18424406E-04
NEW=.20      VAL= 0.23289832E-05
NEW=.30      VAL=-0.22723552E-04
NEW=.40      VAL=-0.50032888E-04
NEW=.50      VAL=-0.77102131E-04
NEW=.60      VAL=-0.10293597E-03
NEW=.70      VAL=-0.12716571E-03
NEW=.80      VAL=-0.14970335E-03
NEW=.90      VAL=-0.17058911E-03
            C= .40
NEW=.10      VAL= 0.44213741E-04
NEW=.20      VAL= 0.44717487E-05
NEW=.30      VAL=-0.56120507E-04
NEW=.40      VAL=-0.12152418E-03
NEW=.50      VAL=-0.18591906E-03
NEW=.60      VAL=-0.24706273E-03
NEW=.70      VAL=-0.30417667E-03
NEW=.80      VAL=-0.35712363E-03
NEW=.90      VAL=-0.40605068E-03
            C= .50
NEW=.10      VAL= 0.87958944E-04
NEW=.20      VAL= 0.57013290E-05
NEW=.30      VAL=-0.11686856E-03
NEW=.40      VAL=-0.24778797E-03
```

```
NEW=.50          VAL=-0.37578498E-03
NEW=.60          VAL=-0.49668048E-03
NEW=.70          VAL=-0.60913697E-03
NEW=.80          VAL=-0.71303151E-03
NEW=.90          VAL=-0.80876152E-03
               C= .60
NEW=.10          VAL= 0.15764551E-03
NEW=.20          VAL= 0.35996449E-05
NEW=.30          VAL=-0.22074958E-03
NEW=.40          VAL=-0.45791240E-03
NEW=.50          VAL=-0.68819765E-03
NEW=.60          VAL=-0.90460024E-03
NEW=.70          VAL=-0.11050897E-02
NEW=.80          VAL=-0.12897078E-02
NEW=.90          VAL=-0.14593511E-02
               C= .70
NEW=.10          VAL= 0.26827675E-03
NEW=.20          VAL=-0.59061941E-05
NEW=.30          VAL=-0.39660064E-03
NEW=.40          VAL=-0.80555772E-03
NEW=.50          VAL=-0.12000921E-02
NEW=.60          VAL=-0.15690713E-02
NEW=.70          VAL=-0.19096366E-02
NEW=.80          VAL=-0.22222855E-02
NEW=.90          VAL=-0.25088439E-02
               C= .80
NEW=.10          VAL= 0.45594367E-03
NEW=.20          VAL=-0.31147512E-04
NEW=.30          VAL=-0.71118113E-03
NEW=.40          VAL=-0.14164509E-02
NEW=.50          VAL=-0.20927498E-02
NEW=.60          VAL=-0.27224388E-02
NEW=.70          VAL=-0.33016243E-02
NEW=.80          VAL=-0.38318410E-02
NEW=.90          VAL=-0.43166766E-02
               C= .90
NEW=.10          VAL= 0.85529485E-03
NEW=.20          VAL=-0.98651250E-04
NEW=.30          VAL=-0.14053942E-02
NEW=.40          VAL=-0.27489950E-02
NEW=.50          VAL=-0.40301967E-02
NEW=.60          VAL=-0.52182089E-02
NEW=.70          VAL=-0.63074524E-02
NEW=.80          VAL=-0.73020332E-02
NEW=.90          VAL=-0.82095449E-02
```

APPENDIX B

APPROXIMATE CALCULATED VALUES OF THE X-FUNCTIONS

APPENDIX C

COMPUTER PROGRAM AND ROOTS OF ORTHOGONAL POLYNOMIALS $C_1(\mu)$

```
C   This program finds the zeros of ORTHOGONAL POLYNOMIALS
C   of degree  N=1 to 5
          IMPLICIT DOUBLE PRECISION (A-H,O-Z)
          DIMENSION P(100),R2(100,100),AM(100,100),AIV(0:100)
          DIMENSION ALP(100),BETA(100)
          DIMENSION C(12),ANU(12),ZZ(100),W12(12)
          READ(21,*)N
          WRITE(24,161)
161       FORMAT(/,32X,'SOLUTION OF ORTHOGONAL POLY EQS',/)
          WRITE(24,181) N
181       FORMAT(33X,'DEGREE OF POLYNOMIAL=',I3,/)
          READ(21,*)(C(I),I=1,10)
          READ(21,*)(ANU(I),I=1,10)
          READ(21,*)(W12(I),I=1,10)

          DO 10 I=1,10
          WRITE(24,171) C(I),ANU(I)
171       FORMAT(/,25X,'C=',F5.2,3X,'NEWO=',D)
          CS=C(I)/(2.D0*W12(I)*(1.D0-C(I)))
          CR=CS*ANU(I)*DSQRT(1.D0-C(I))
          DO 20 II=0,22
          IF(II.EQ.0) GO TO 21
          XX=FLOAT(II)
          AIV(II)=(1.0D0-XX*ANU(I)*AIV(II-1))/XX
          GO TO 20
21        AIV(II)=DLOG((ANU(I)+1.D0)/ANU(I))
20        CONTINUE
          DO 30 IR=1,11
          DO 30 IC=1,11
30        AM(IR,IC)=CR*AIV(IR+IC-1)+CS*AIV(IR+IC)
          R2(1,1)=DSQRT(AM(1,1))
          DO 40 IC1=2,11
40        R2(1,IC1)=AM(1,IC1)/R2(1,1)
          DO 50 IR=2,11
          DO 50 IC=IR,11
          IF(IR.EQ.IC) GO TO 51
          SIGMA=0.0D0
          DO 52 IS=1,IR-1
52        SIGMA=SIGMA+R2(IS,IR)*R2(IS,IC)
          R2(IR,IC)=(AM(IR,IC)-SIGMA)/R2(IR,IR)
          GO TO 50
51        SIGMA=0.0D0
          DO 53 IS=1,IR-1
53        SIGMA=SIGMA+R2(IS,IR)**2
          R2(IR,IR)=DSQRT(AM(IR,IR)-SIGMA)
50        CONTINUE
          ALP(1)=R2(1,2)/R2(1,1)
          BETA(1)=R2(2,2)/R2(1,1)
          DO 60 J=2,N
          ALP(J)=(R2(J,J+1)/R2(J,J))-(R2(J-1,J)/R2(J-1,J-1))
          IF(J.EQ.N) GO TO 60
          BETA(J)=R2(J+1,J+1)/R2(J,J)
60        CONTINUE
          CALL ROOTS(N,ALP,BETA,ZZ)
10        CONTINUE
          STOP;END
C         TO FIND ROOTS OF POLYNOMIALS BY HALF-INTERVAL METHOD
```

```
      SUBROUTINE ROOTS(N,ALP,BETA,ZZ)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION P(0:100),ZZ(100)
      DIMENSION ALP(100),BETA(100)
      NP1=N
      ITER=DLOG(0.05D0/1.0D-12)/DLOG(2.0D0)+1.0D0
C     TO ESTABLISH INTERVAL WITHIN WHICH ROOT LIES
      ZR=0.0D0
      DO 7 I=1,NP1
      ZL=ZR
1     IF((POLY(ZL,ALP,BETA,N)*POLY(ZL+0.05D0,ALP,BETA,N)).LT.(
     1GO TO 3
      ZL=ZL+0.05D0
      If(ZL.Gt.1.0D0) GO tO 44
      GO TO 1
3     ZR=ZL+0.05D0
      FZL=POLY(ZL,ALP,BETA,N)
C     BEGIN HALF-INTERVAL ITERATION
      DO 6 J=1,ITER
      ZHALF=(ZL+ZR)/2.0D0
      FZHALF=POLY(ZHALF,ALP,BETA,N)
C     XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
C     CHOOSE THE SUB-INTERVAL CONTAING THE ROOT...
      IF(FZHALF*FZL.LE.0.0D0) GO TO 5
      ZL=ZHALF
      FZL=FZHALF
      GO TO 6
5     ZR=ZHALF
6     CONTINUE
      ZZ(I)=(ZL+ZR)/2.0D0
      WRITE(24,191) ZZ(I)
191   FORMAT(30X,'MEW=',D)
      FUNVAL=POLY(ZZ(I),ALP,BETA,N)
      IF(ZZ(I).GT.1.0D0) GO TO 44
7     CONtINUE
44    FRAUD=FRAUD
      RETURN
      END
      DOUBLE PRECISION FUNCTION POLY(X,ALP,BETA,N)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION P(0:100)
      DIMENSION ALP(100),BETA(100)
      P(0)=1.0D0
      P(1)=(X-ALP(1))/BETA(1)
      IF(N.EQ.1) GO TO 33
      DO 70 J=2,N
      P(J)=((X-ALP(J))*P(J-1)-BETA(J-1)*P(J-2))/BETA(J)
70    CONTINUE
33    VAL=P(N)
      POLY=VAL
      RETURN
      END
```

SOLUTION OF ORTHOGONAL POLY EQS

DEGREE OF POLYNOMIAL= 1

```
C= 0.10    NEWO=  0.100000000410000000D+01
     MEW=  0.667876997685016250D+00

C= 0.20    NEWO=  0.100009088650000000D+01
     MEW=  0.669245241277894821D+00

C= 0.30    NEWO=  0.100259288800000000D+01
     MEW=  0.670810504576729728D+00

C= 0.40    NEWO=  0.101458581600000000D+01
     MEW=  0.672619457995097037D+00

C= 0.50    NEWO=  0.104438203400000000D+01
     MEW=  0.674736589049643954D+00

C= 0.60    NEWO=  0.110213202200000000D+01
     MEW=  0.677267614627271542D+00

C= 0.70    NEWO=  0.120680425800000000D+01
     MEW=  0.680403117266178016D+00

C= 0.80    NEWO=  0.140763430100000000D+01
     MEW=  0.684536323020074634D+00

C= 0.90    NEWO=  0.190320484300000000D+01
     MEW=  0.690738592815978337D+00
```

SOLUTION OF ORTHOGONAL POLY EQS

DEGREE OF POLYNOMIAL= 2

```
C= 0.10    NEWO=  0.1000000004100000D+01
      MEW=  0.35609981109264814D+00
      MEW=  0.84538644571257464 3D+00

C= 0.20    NEWO=  0.1000090886500000D+01
      MEW=  0.35730260063610330 7D+00
      MEW=  0.84588039108675730 1D+00

C= 0.30    NEWO=  0.1002592888000000D+01
      MEW=  0.35870012234408932 3D+00
      MEW=  0.84644493933556077 7D+00

C= 0.40    NEWO=  0.1014585816000000D+01
      MEW=  0.36033766083310183 7D+00
      MEW=  0.84709746529733820 5D+00

C= 0.50    NEWO=  0.1044382034000000D+01
      MEW=  0.36227311842121707 8D+00
      MEW=  0.84786244695751520 5D+00

C= 0.60    NEWO=  0.1102132022000000D+01
      MEW=  0.36459937225263274 8D+00
      MEW=  0.84878005845348525 5D+00

C= 0.70    NEWO=  0.1206804258000000D+01
      MEW=  0.36748308416317741 0D+00
      MEW=  0.84992275633740064 4D+00

C= 0.80    NEWO=  0.1407634301000000D+01
      MEW=  0.37126609428341908 0D+00
      MEW=  0.85144059536287386 6D+00

C= 0.90    NEWO=  0.1903204843000000D+01
      MEW=  0.37687016821982979 4D+00
      MEW=  0.85374527991043578 5D+00
```

```
         SOLUTION OF ORTHOGONAL POLY EQS

          DEGREE OF POLYNOMIAL=  3

C= 0.10    NEWO=   0.10000000041000000D+01
      MEW=  0.21298866734032344 6D+00
      MEW=  0.59124223859726044 0D+00
      MEW=  0.91160836956223647 8D+00

C= 0.20    NEWO=   0.10000908865000000D+01
      MEW=  0.21373753298285009 8D+00
      MEW=  0.59204905059123120 9D+00
      MEW=  0.91183008722837257 9D+00

C= 0.30    NEWO=   0.10025928880000000D+01
      MEW=  0.21461475319702003 7D+00
      MEW=  0.59297871338822006 0D+00
      MEW=  0.91208359094198386 0D+00

C= 0.40    NEWO=   0.10145858160000000D+01
      MEW=  0.21564997753848729 2D+00
      MEW=  0.59406065762486832 8D+00
      MEW=  0.91237684131228889 1D+00

C= 0.50    NEWO=   0.10443820340000000D+01
      MEW=  0.21687933466128015 5D+00
      MEW=  0.59533455625460192 2D+00
      MEW=  0.91272113687118590 2D+00

C= 0.60    NEWO=   0.11021320220000000D+01
      MEW=  0.21835971021318982 9D+00
      MEW=  0.59686490795102145 2D+00
      MEW=  0.91313504039244435 5D+00

C= 0.70    NEWO=   0.12068042580000000D+01
      MEW=  0.22019252280333603 3D+00
      MEW=  0.59876780127997335 7D+00
      MEW=  0.91365207062590343 6D+00

C= 0.80    NEWO=   0.14076343010000000D+01
      MEW=  0.22258447015883575 7D+00
      MEW=  0.60128256059651903 5D+00
      MEW=  0.91434183501660300 2D+00

C= 0.90    NEWO=   0.19032048430000000D+01
      MEW=  0.22608794108709844 3D+00
      MEW=  0.60506031757722667 1D+00
      MEW=  0.91539617482012545 5D+00
```

SOLUTION OF ORTHOGONAL POLY EQS

DEGREE OF POLYNOMIAL= 4

```
C= 0.10    NEWO=  0.1000000004100000000D+01
       MEW= 0.1401580618392472390D+00
       MEW= 0.4170532671527325880D+00
       MEW= 0.7235910629537102080D+00
       MEW= 0.9429991283086565120D+00

C= 0.20    NEWO=  0.1000090886500000000D+01
       MEW= 0.1406199551962345140D+00
       MEW= 0.4177901417566317840D+00
       MEW= 0.7240835772219725200D+00
       MEW= 0.9431158498136937880D+00

C= 0.30    NEWO=  0.1002592888000000000D+01
       MEW= 0.1411633422289014560D+00
       MEW= 0.4186447407100786230D+00
       MEW= 0.7246494792434532450D+00
       MEW= 0.9432493593169056140D+00

C= 0.40    NEWO=  0.1014585816000000000D+01
       MEW= 0.1418069446390290980D+00
       MEW= 0.4196446914644184300D+00
       MEW= 0.7253067616959015160D+00
       MEW= 0.9434039015959569980D+00

C= 0.50    NEWO=  0.1044382034000000000D+01
       MEW= 0.1425728857608191900D+00
       MEW= 0.4208257379290444080D+00
       MEW= 0.7260802633270941460D+00
       MEW= 0.9435855242754769310D+00

C= 0.60    NEWO=  0.1102132022000000000D+01
       MEW= 0.1434955406042718100D+00
       MEW= 0.4222453360383951810D+00
       MEW= 0.7270105676030652850D+00
       MEW= 0.9438041710283869210D+00

C= 0.70    NEWO=  0.1206804258000000000D+01
       MEW= 0.1446358850942488060D+00
       MEW= 0.4240065454014256840D+00
       MEW= 0.7281708432401501360D+00
       MEW= 0.9440778142943600000D+00

C= 0.80    NEWO=  0.1407634301000000000D+01
       MEW= 0.1461176133027038310D+00
       MEW= 0.4263207460109697430D+00
       MEW= 0.7297124860084295520D+00
       MEW= 0.9444438452246686240D+00

C= 0.90    NEWO=  0.1903204843000000000D+01
       MEW= 0.1482692791880253940D+00
       MEW= 0.4297582142960891370D+00
       MEW= 0.7320499983074114440D+00
       MEW= 0.9450056097553897450D+00
```

```
          SOLUTION OF ORTHOGONAL POLY EQS
           DEGREE OF POLYNOMIAL=  5

C= 0.10     NEWO=   0.10000000041000000000D+01
         MEW=  0.98790051682954072 2D-01
         MEW=  0.30504509007841989 0D+00
         MEW=  0.56251753937140165 6D+00
         MEW=  0.80226118088939984 2D+00
         MEW=  0.96025082243686484 1D+00

C= 0.20     NEWO=   0.10000908865000000000D+01
         MEW=  0.99086440258906805 1D-01
         MEW=  0.30563080837564484 8D+00
         MEW=  0.56307867170071404 1D+00
         MEW=  0.80257236626166559 6D+00
         MEW=  0.96031938825653924 5D+00

C= 0.30     NEWO=   0.10025928880000000000D+01
         MEW=  0.99435944463402847 9D-01
         MEW=  0.30631337623672152 5D+00
         MEW=  0.56372644865368784 1D+00
         MEW=  0.80292948707974574 3D+00
         MEW=  0.96039784258937288 9D+00

C= 0.40     NEWO=   0.10145858160000000000D+01
         MEW=  0.99850700773458811 6D-01
         MEW=  0.30711521167941100 5D+00
         MEW=  0.56448166740701708 7D+00
         MEW=  0.80334397210353927 6D+00
         MEW=  0.96048870064078073 9D+00

C= 0.50     NEWO=   0.10443820340000000000D+01
         MEW=  0.10034475683387427 1D+00
         MEW=  0.30806435733866237 6D+00
         MEW=  0.56537217469049210 4D+00
         MEW=  0.80383179686577932 8D+00
         MEW=  0.96059555396168435 0D+00

C= 0.60     NEWO=   0.11021320220000000000D+01
         MEW=  0.10093973909642954 8D+00
         MEW=  0.30920539735707279 8D+00
         MEW=  0.56644313674369186 8D+00
         MEW=  0.80441913079375808 9D+00
         MEW=  0.96072431120664987 7D+00

C= 0.70     NEWO=   0.12068042580000000000D+01
         MEW=  0.10167387440415041 1D+00
         MEW=  0.31061801381329132 8D+00
         MEW=  0.56777578778819588 7D+00
         MEW=  0.80515321396705985 8D+00
         MEW=  0.96088565935824590 7D+00

C= 0.80     NEWO=   0.14076343010000000000D+01
         MEW=  0.10262446436281607 0D+00
         MEW=  0.31246491677047743 1D+00
         MEW=  0.56953747446568741 1D+00
         MEW=  0.80613204367509752 3D+00
         MEW=  0.96110186032638012 3D+00

C= 0.90     NEWO=   0.19032048430000000000D+01
         MEW=  0.10399601218487077 9D+00
         MEW=  0.31518212434930319 5D+00
         MEW=  0.57218295498496445 5D+00
         MEW=  0.80762507072176959 2D+00
         MEW=  0.96143456183162925 7D+00
```

BIBLIOGRAPHY

1. CASE K M and ZWEIFEL P F, Linear Transport Theory, (Reading, Mass: Addison Wesley), (1967).

2. GANGULY K and SENGUPTA A, A Transport Theoretic P_N Approximation, Nucl. Sic. Engng, 74, 1 (1980); Interface and Source Problems by a Transport Theoretic P_N Approximation, Nucl. Sci. Engng, 77, 13 (1981).

3. SENGUPTA A, A Transport Theoretic Approximation Method for the Monoenergetic Neutron Transport Equation, J. Phys. A: Math. Gen. 15, L487-L493, (1982).

4. SENGUPTA A, A Rational Function Approximation of the Singular Eigen-Function of the Monoenergetic Neutron Transport Equation, J. Phys. A. Math. Gen., 17 2743-2758 (1984).

5. SENGUPTA A, A New Set of Orthogonal Polynomials for use in Neutron Transport Theory, Submitted for Publication, April 1985.

6. SIEWERT C E and BENOIST P, The F_N Method in Neutron-Transport Theory Part I: Theory and Applications, Nucl. Sci. Engng. 69, 156-160 (1975).

7. SIEWERT C E and GRANOJEAN P, The F_N Method in Neutron-Transport Theory Part II: Applications and Numerical Results; Nucl. Sci. Engng., 69, 161-168 (1979).

8. SENGUPTA A, On the Approximate Solution of a Class of Singular Integral Equations, Submitted for Publication February 1985.